ON LINEAR FORMS WHOSE COEFFICIENTS ARE IN ARITHMETIC PROGRESSION

BY

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ABSTRACT

Let Ω be the set of positive integers that are omitted values of the form $f = \Sigma_{i=1}^{n} a_i x_i$ where the a_i are fixed positive integers with g. c. d. 1 and the x_i are variable nonnegative integers.
Set $\omega = |\Omega|$ and $\kappa = \max \Omega + 1$. Using an expression of Roberts [4] for κ when the a_i form an arithmetic progression, we determine ω in this case.

1. Introduction

Let $a_1, a_2, \cdots, a_n$ be positive integers with g.c.d. 1. As $x_1, x_2, \cdots, x_n$ run independently over the nonnegative integers, the values of the linear form

$$f = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \tag{1}$$

run over a set of nonnegative integers. As pointed out by Nijenhuis and Wilf [3], this set of assumed values, together with the operation of addition, forms a semigroup.

Schur (see Brauer [2]), showed that there exists a positive integer m_0 such that all $m \geqq m_0$ are assumed by f.

Let $\kappa(f)$, the *conductor* of f, be the least positive integer m_0 for which f assumes all integers $\geqq m_0$.

Let $\Omega = \Omega(f)$ be the set of positive integers that are omitted values of f, and let $\omega(f) = |\Omega|$. Clearly $\kappa(f) = \max \Omega + 1$.

It has been known for a long time that if $n = 2$, then

$$\kappa(f) = (a_1 - 1)(a_2 - 1) \tag{2}$$

and a classical result of Sylvester [5] shows that

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(3) $$\omega(f) = \tfrac{1}{2}(a_1 - 1)(a_2 - 1) = \tfrac{1}{2}\kappa(f).$$

Simple proofs of (2) and (3) are presented in Nijenhuis and Wilf [3].

It is the purpose of this paper to evaluate $\omega(f)$ when f is the form

(4) $$ax_0 + (a + d)x_1 + \cdots + (a + sd)x_s.$$

Denote this form by ϕ. When we want to emphasise the s-dependence of ϕ, we shall use the notation ϕ_s.

2. Preliminary results

We require some lemmas which we state without proof.

LEMMA 1 (Roberts [4]). *The only numbers assumed by the form* ϕ *with* $\sum_{i=0}^{s} x_i = m$ *are* ma, $ma + d, \cdots, ma + msd$.

LEMMA 2. *If* $(a, d) = 1$, *then*

(5) $$\sum_{k=1}^{d-1} \left[\frac{ka}{d}\right] = \tfrac{1}{2}(a - 1)(d - 1).$$

Here $[x]$ denotes the greatest integer $\leqq x$.

LEMMA 3 (Roberts [4], Bateman [1]).

(6) $$\kappa(\phi) = \left(\left[\frac{a-2}{s}\right] + 1\right)a + (a - 1)(d - 1).$$

LEMMA 4 (Nijenhuis and Wilf [3]).

For the general form (1),

(7) $$\omega(f) \geqq \tfrac{1}{2}\kappa(f).$$

3. Main theorem

Let $\bar{\Omega} = \bar{\Omega}(\phi)$ be the set of integers $< \kappa(\phi)$ which are assumed by ϕ. Let $\bar{\omega}(\phi) = |\bar{\Omega}|$.

Clearly

(8) $$\omega(\phi) + \bar{\omega}(\phi) = \kappa(\phi).$$

THEOREM.

(9) $$\omega(\phi) = \tfrac{1}{2}\left(\left[\frac{a-2}{s}\right] + 1\right)(a + t) + \tfrac{1}{2}(a - 1)(d - 1)$$

where t *is the smallest nonnegative integer for which*

(10) $$a - 2 \equiv t (\bmod s).$$

PROOF. If $a = 1$, the result is trivial. Assume $a > 1$. By Lemma 1, we can classify the integers assumed by ϕ by considering in turn all possible values of $\Sigma_{i=0}^{s} x_i$. We obtain the following sets of assumed values, with G_n containing all those obtained with $\Sigma_{i=0}^{s} x_i = n$:

$$\begin{aligned}
G_0 &= \{0\} \\
G_1 &= \{a, a + d, \cdots, a + sd\} \\
&\vdots \\
G_n &= \{na, na + d, \cdots, na + nsd\}. \\
&\vdots
\end{aligned}$$

Let

(11) $$v_n = |G_n|; \text{ clearly } v_n = ns + 1.$$

We have

(12) $$\bar{\omega}(\phi) = \sum_{i=0}^{\alpha} |G_i \cap \bar{\Omega}|$$

where the upper terminal α is such that $|G_{\alpha+1} \cap \bar{\Omega}| = 0$.

Provided the largest term in G_i is small enough, $G_i \subset \bar{\Omega}$.

Let $N = \max\{n: G_n \subset \bar{\Omega}\}$. Thus for all $i \leqq N$, $G_i \subset \bar{\Omega}$.

Let

(13) $$\bar{\omega}_1 = \sum_{i=0}^{N} |G_i \cap \bar{\Omega}| = \sum_{i=0}^{N} v_i.$$

Set

(14) $$g = \kappa(\phi) - 1, \; m = a + sd \text{ and } A = \left[\frac{(a-2)}{s}\right].$$

It follows from the definition of N that $N = [g/m]$. We show that $[g/m] = A$.

We have:

$$A < \frac{(a-1)}{s} \leqq \left[\frac{(a-1)}{s} + \frac{(s-1)}{s}\right] = A + 1$$

so

$$sdA < (a-1)d \leqq sd(A+1) < a + sd(A+1).$$

Adding Aa, we obtain

$$(a+sd)A < Aa + (a-1)d < (a+sd)(A+1)$$

i.e.

$$mA < g < m(A+1).$$

So

$$A < \frac{g}{m} < A+1, \text{ whence } \left[\frac{g}{m}\right] = A. \text{ Thus } N = A.$$

Equation (13) then gives $\bar{\omega}_1 = \sum_{i=0}^{A} (is+1) = \frac{1}{2}(A+1)(As+2)$.

Since $As + 2 \equiv 2 \pmod{s}$, we have

$$As + 2 = \left[\frac{(a-2)}{s}\right]s + 2 = a - t.$$

Thus

$$\bar{\omega}_1 = \tfrac{1}{2}\left(\left[\frac{a-2}{s}\right] + 1\right)(a-t). \tag{15}$$

The remaining elements of $\bar{\Omega}$ are those in $G_i \cap \bar{\Omega}$ with $i > N$.

Let

$$\bar{\omega}_2 = \sum_{k=N+1}^{\alpha} |G_k \cap \bar{\Omega}| = \sum_{k=N+1}^{\alpha} v'_k. \tag{16}$$

Clearly

$$\bar{\omega}(\phi) = \bar{\omega}_1 + \bar{\omega}_2. \tag{17}$$

Now $|G_{N+i} \cap \bar{\Omega}| \neq 0$ for a positive integer i if and only if there is a nonnegative integer k_i such that:

$$(N+i)a + k_i d < g < (N+i)a + (k_i+1)d.$$

Then

$$k_i < \frac{g-(N+i)a}{d} < k_i + 1, \text{ and hence}$$

$$k_i = \left[\frac{g-(N+i)a}{d}\right] = \left[\frac{g-(A+i)a}{d}\right] = \left[a - 1 - \frac{ia}{d}\right].$$

Clearly

$$v'_{N+i} = k_i + 1 = \left[a - \frac{ia}{d}\right]. \tag{18}$$

Now

$$v'_{N+d} = \left[a - \frac{da}{d}\right] = 0.$$

So

$$\bar{\omega}_2 = \sum_{i=1}^{d-1} \left[a - \frac{ia}{d}\right] = \sum_{i=1}^{d-1} \left[\frac{a}{d}(d-i)\right] = \sum_{k=1}^{d-1} \left[\frac{ka}{d}\right] = \tfrac{1}{2}(a-1)(d-1) \tag{19}$$

by Lemma 2.

Thus

$$\bar{\omega}(\phi) = \bar{\omega}_1 + \bar{\omega}_2 = \tfrac{1}{2}\left(\left[\frac{a-2}{s}\right] + 1\right)(a-t) + \tfrac{1}{2}(a-1)(d-1). \tag{20}$$

From (8), we may deduce that:

$$\omega(\phi) = \tfrac{1}{2}\left(\left[\frac{a-2}{s}\right] + 1\right)(a+t) + \tfrac{1}{2}(a-1)(d-1)$$

which is the result required.

REMARK. It is interesting to note that $\bar{\omega}_1$ is independent of d, whereas ω_2 is independent of s.

There are three corollaries.

COROLLARY 1.

$\omega(\phi) = \tfrac{1}{2}\kappa(\phi)$ if and only if $a \equiv 2 \pmod{s}$.

COROLLARY 2. *In the case $s=1$, the congruence $a \equiv 2 \pmod{s}$ holds trivially; thus in this case Corollary 1 gives*

$$\omega(\phi_1) = \tfrac{1}{2}\kappa(\phi_1)$$

which is the result of Sylvester [5].

COROLLARY 3. *For given a and d we minimise $\omega(\phi_s)$ by taking $s = a-1$. In this case,*

$$\omega(\phi_{a-1}) = \tfrac{1}{2}(a-1)(d+1). \tag{21}$$

PROOF. The sequence $\{\omega(\phi_s)\}$ is nonincreasing with s. For all $s \geqq a-1$, we have $\omega(\phi_s) = \tfrac{1}{2}(a+t) + \tfrac{1}{2}(a-1)(d-1)$.

As $t = a - 2$ in this case, we have $\omega(\phi_{a-1}) = \frac{1}{2}(a-1)(d+1)$.
For all $s \geqq a - 1$ we also have $\omega(\phi_s) = \frac{1}{2}(a-1)(d+1)$.

Now $\omega(\phi_{a-2}) = \frac{1}{2}(a-1)(d+1) + 1 > \omega(\phi_{a-1})$ as required.

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